ITERATED RIESZ COMMUTATORS: A SIMPLE PROOF OF BOUNDEDNESS

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Abstract. We give a simple proof of L^p boundedness of iterated commutators of Riesz transforms and a product BMO function. We use a representation of the Riesz transforms by means of simple dyadic operators - dyadic shifts - which in turn reduces the estimate quickly to paraproduct estimates.

1. Introduction

It was shown by the authors in [14] that product BMO of S.-Y. A Chang and R. Fefferman, defined on the space $\mathbb{R}^{d_1} \otimes \cdots \otimes \mathbb{R}^{d_t}$, can be characterized by the iterated commutators of Riesz transforms in the multi-parameter setting. This extended a classical one-parameter result of R. Coifman, R. Rochberg, and G. Weiss [3], and at the same time extended the work of M. Lacey and S. Ferguson [7] and M. Lacey and E. Terwilleger [13], which treats the case of iterated commutators with Hilbert transforms.

In the multi parameter setting, one seeks to prove an inequality of the type

$$\|b\|_{BMO} \overset{(1)}{\lesssim} \|[T_1,[T_2,...[T_n,M_b]...]]\|_{L^2\to L^2} \overset{(2)}{\lesssim} \|b\|_{BMO}$$

where T_i are either Riesz or Hilbert transforms acting in the ith variable and M_b is multiplication by b. We refer to (1) as a lower bound and (2) as an upper bound.

In the case of the Hilbert transforms, the upper bound is not hard to prove, by appealing to analyticity even in the multi parameter setting, see [8], whereas the lower bound is extremely difficult and makes use of the corresponding upper bound. Similar considerations for the Riesz transforms fail because of a lack of analyticity. The upper bound is already difficult, even in the case of one parameter, and the proof of the lower bound is non-trivial. Coifman, Rochberg and Weiss [3] used a sharp function method to prove their one-parameter analog of

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Theorem 1.1. This method, versatile as it is in the one-parameter case, has no clear analog in the product setting; see Lacey [12] for a discussion of this in a similar context. The methods of [3] do not extend to the multi parameter setting.

In [14] we find it necessary to prove an upper bound for more generalized smooth cone projection operators to simulate the analyticity we do not have. Such operators, even in the one parameter case, are not included in the work in [3]. The upper bound for the Riesz transforms in particular allow for a relatively simple proof that extends naturally to the multi parameter setting. In this note, we present this proof. Here is the theorem:

1.1. **Theorem.**

$$\|[R_{1,j_1},[R_{2,j_2},...[R_{n,j_n},M_b]...]]\|_{L^2\to L^2} \lesssim \|b\|_{BMO}$$

Here $R_{s,\,j}$ denotes the jth Riesz transform acting on the sth variable on $\mathbb{R}^{d_s}$ and M_b denotes multiplication by the symbol b. The BMO norm used is Cheng Fefferman product BMO.

The proof given here is rather different from that of Coifman, Rochberg and Weiss. The proof of the upper bound presented here follows from the decomposition of the commutators into a sum of simpler terms. The decomposition we allude to is naturally described in terms of 'Haar shifts,' a method of proof used by Petermichl [18] to address a subtle question concerning the Hilbert transform in a context with matrix valued weights and then extended by Petermichl, Treil and Volberg in [19] to the Riesz transforms. This method reduces the main issue to paraproduct estimates quickly. The paraproducts that arise are of multi-parameter form. The specific result needed is Theorem 5.1 below. This result is due to Journé [9]; more recent discussions of paraproducts are in [11, 16, 17].

2. The Upper Bound

Let $\mathrm{M}_b\,\varphi \stackrel{\text{def}}{=} b\varphi$ be the operator of pointwise multiplication by a function b. For Schwartz functions f on $\mathbb{R}^d$, let $\mathrm{R}_j\, f$ denote the jth Riesz transform of f, for $1 \le j \le d$.

We are concerned with product spaces $\mathbb{R}^{\vec d} = \mathbb{R}^{d_1}\otimes\cdots\otimes\mathbb{R}^{d_t}$ for vectors $\vec d = (d_1,\dots,d_t) \in \mathbb{N}^t$. For Schwartz functions b, f on $\mathbb{R}^{\vec d}$, and for a vector $\vec j = (j_1,\dots,j_t)$ with $1 \le j_s \le d_s$ for $s = 1,\dots,t$ we consider the family of commutators

$$\mathrm{C}_{\vec j}(b,f)(x) \stackrel{\text{def}}{=} [\cdots[[\mathrm{M}_b,\mathrm{R}_{1,\,j_1}],\mathrm{R}_{2,\,j_2}],\cdots](f)(x) \tag{2.1}$$

where $\mathrm{R}_{s,\,j}$ denotes the jth Riesz transform acting on $\mathbb{R}^{d_s}$.

2.2. **Theorem.** *We have the estimates below, valid for* $1 < p < \infty$.

$$\|\mathrm{C}_{\vec j}(b,\varphi)\|_p \lesssim \|b\|_{\mathrm{BMO}}\|\varphi\|_p\,. \tag{2.3}$$

By BMO, *we mean Chang–Fefferman* BMO.

3. Haar Functions in Several Dimensions

We start in the one-parameter setting. We will use dilation and translation operators on $\mathbb{R}^d$

$$\operatorname{Tr}_y f(x) \stackrel{\text{def}}{=} f(x-y)\,, \quad y \in \mathbb{R}^d, \tag{3.1}$$

$$\operatorname{Dil}_a^{(p)} f(x) \stackrel{\text{def}}{=} a^{-d/p} f(x/a)\,, \quad a>0\,,\ 0<p\le\infty\,. \tag{3.2}$$

These will also be applied to sets, in an obvious fashion, in the case of $p=\infty$.

By the *canonical dyadic grid* in $\mathbb{R}^d$ we mean the collection of cubes

$$\mathcal{D}_{\text{cncl}} \stackrel{\text{def}}{=} \left\{ j2^k + [0,2^k)^d \,:\, j\in\mathbb{Z}^d\,,\ k\in\mathbb{Z} \right\}$$

By a *dyadic grid* we mean any of the collections

$$\mathcal{D}_d = \mathcal{D}_d^{t,y} = \left\{ \operatorname{Dil}_t^{(\infty)} \operatorname{Tr}_y I \,:\, I\in\mathcal{D}_{\text{cncl}} \right\}, \qquad 1\le t\le 2\,,\ y\in\mathbb{R}^d\,.$$

Note that $\mathcal{D}_{\text{cncl}} = \mathcal{D}_d^{1,0}$. If we emphasize the role of dimension, we refer to these collections as *d-dimensional dyadic grids.*

Haar functions on $\mathbb{R}^d$ are now described. For $\varepsilon\in\{0,1\}$, set

$$h^0 \stackrel{\text{def}}{=} -\mathbf{1}_{[-\frac12,0)} + \mathbf{1}_{[0,\frac12)}\,, \qquad h^1 \stackrel{\text{def}}{=} \mathbf{1}_{(-\frac12,\frac12)}\,.$$

Here, we put the superscript ^{0} to denote that 'the function has mean 0,' while a superscript ^{1} denotes that 'the function is an L^2 normalized indicator function.' In one dimension, for an interval I, set

$$h_I^\varepsilon \stackrel{\text{def}}{=} \operatorname{Tr}_{c(I)} \operatorname{Dil}_{|I|}^{(2)} h^\varepsilon\,.$$

Of course for any choice of one dimensional grid $\mathcal{D}$, the collections of functions $\{h_I^0 \,:\, I\in\mathcal{D}\}$ form a Haar basis for $L^p(\mathbb{R})$.

Let $\operatorname{Sig}_d \stackrel{\text{def}}{=} \{0,1\}^d - \{\vec{1}\}$, which we refer to as *signatures*. In d dimensions, for a cube Q with side I, i.e., $Q = I\times\cdots\times I$, and a choice of $\varepsilon\in\operatorname{Sig}_d$, set

$$h_Q^\varepsilon(x_1,\dots,x_d) \stackrel{\text{def}}{=} \prod_{j=1}^d h_I^{\varepsilon_j}(x_j).$$

It is then the case that the collection of functions

$$\operatorname{Haar}_{\mathcal{D}_d} \stackrel{\text{def}}{=} \{h_Q^\varepsilon \,:\, Q\in\mathcal{D}_d\,,\ \varepsilon\in\operatorname{Sig}_d\}$$

form a Haar basis for $L^p(\mathbb{R}^d)$ for any choice of d-dimensional dyadic grid $\mathcal{D}_d$. Here, we are using the notation $\vec{1} = (1,\dots,1)$. While we exclude the superscript $\vec{1}$ here, they play a role in the theory of paraproducts.

We will use these bases in the tensor product setting. Thus, for a vector $\vec{d} = (d_1, \dots, d_t)$, and $1 \le s \le t$, let $\mathcal{D}_{d_s}$ be a choice of d_s dimensional dyadic grid, and let

$$\mathcal{D}_{\vec{d}} = \otimes_{s=1}^{t} \mathcal{D}_{d_s} .$$

Also, let $\operatorname{Sig}_{\vec{d}} \stackrel{\text{def}}{=} \{\vec{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_t) : \varepsilon_s \in \operatorname{Sig}_{d_s}\}$. Note that each ε_s is a vector, and so $\vec{\varepsilon}$ is a 'vector of vectors'. For a rectangle $R = Q_1 \times \cdots Q_t$, i.e. a product of cubes of possibly different dimensions, and a choice of vectors $\vec{\varepsilon} \in \operatorname{Sig}_{\vec{d}}$ set

$$h_R^{\vec{\varepsilon}}(x_1, \dots, x_t) = \prod_{s=1}^{t} h_{Q_s}^{\varepsilon_s}(x_s)$$

These are the appropriate functions and bases to analyze multi-parameter paraproducts and commutators.

Let

$$\operatorname{Haar}_{\mathcal{D}_{\vec{d}}} \stackrel{\text{def}}{=} \left\{ h_R^{\vec{\varepsilon}} \,:\, R \in \mathcal{D}_{\vec{d}},\ \vec{\varepsilon} \in \operatorname{Sig}_{\vec{d}} \right\} .$$

This is a basis in $L^p(\mathbb{R}^{\vec{d}})$, where we use the notation

$$\mathbb{R}^{\vec{d}} \stackrel{\text{def}}{=} \mathbb{R}^{d_1} \otimes \cdots \otimes \mathbb{R}^{d_t}$$

to emphasize that we are in a tensor product setting.

4. Chang–Fefferman BMO

We describe the elements of product Hardy space theory, as developed by S.-Y. Chang and R. Fefferman [1, 2, 4–6] as well as Journé [9, 10]. By this, we mean the Hardy spaces associated with domains like $\otimes_{s=1}^{t} \mathbb{R}^{d_s}$.

4.1. *Remark.* The (real) Hardy space $H^1(\mathbb{R}^d)$ typically denotes the class of functions with the norm

$$\sum_{j=0}^{d} \| \mathrm{R}_j \, f \|_1$$

where R_j denotes the jth Riesz transform. Here and below we adopt the convention that R_0, the 0th Riesz transform, is the identity. This space is invariant under the one-parameter family of isotropic dilations, while $H^1(\mathbb{R}^{\vec{d}})$ is invariant under dilations of each coordinate separately. That is, it is invariant under a t-parameter family of dilations, hence the terminology 'multiparameter' theory.

As before, the space $H^1(\mathbb{R}^{\vec{d}})$ has a variety of equivalent norms, in terms of square functions, maximal functions and Riesz transforms. For our discussion, the characterization in terms of Riesz transforms is most useful:

$$\|f\|_{H^1(\mathbb{R}^{\vec{d}})} = \sum_{\vec{0}\le\vec{j}\le\vec{d}} \Big\| \prod_{s=1}^{t} \mathrm{R}_{s,j_s}\, f \Big\|_1 .$$

R_{s,j_s} is the Riesz transform computed in the j_sth direction of the sth variable, and the 0th Riesz transform is the identity operator.

4.1. **BMO($\mathbb{R}^{\vec{d}}$).** The dual of the real Hardy space is $H^1(\mathbb{R}^{\vec{d}})^* = \mathrm{BMO}(\mathbb{R}^{\vec{d}})$, the t–fold product BMO space. It is a Theorem of S.-Y. Chang and R. Fefferman [2] that this space has a characterization in terms of a product Carleson measure.

Define

$$\|b\|_{\mathrm{BMO}(\mathbb{R}^{\vec{d}})} \stackrel{\mathrm{def}}{=} \sup_{U\subset\mathbb{R}^{\vec{d}}} \Big[|U|^{-1} \sum_{R\subset U} \sum_{\vec{\varepsilon}\in\mathrm{Sig}_{\vec{d}}} |\langle b, w_R^{\vec{\varepsilon}}\rangle|^2 \Big]^{1/2} . \tag{4.2}$$

Here the supremum is taken over all open subsets $U \subset \mathbb{R}^{\vec{d}}$ with finite measure, and we use a wavelet basis $w_R^{\vec{\varepsilon}}$.

4.3. **Theorem** (Chang–Fefferman BMO)**.** *We have the equivalence of norms*

$$\|f\|_{(H^1(\mathbb{R}^{\vec{d}}))^*} \approx \|f\|_{\mathrm{BMO}(\mathbb{R}^{\vec{d}})}.$$

That is, $\mathrm{BMO}(\mathbb{R}^{\vec{d}})$ *is the dual to* $H^1(\mathbb{R}^{\vec{d}})$*.*

5. Paraproducts

We recall a result of Journé [9], with a recent improvement of Muscalu, Pipher, Tao and Thiele [16, 17]. (Also see Lacey and Metcalfe [11].)

Consider the bilinear operators, in fact multi-parameter paraproducts, on functions in $\mathbb{R}^{\vec{d}}$, given as

$$\mathrm{B}(f_1, f_2) \stackrel{\mathrm{def}}{=} \sum_{R\in\mathcal{D}_{\vec{d}}} \epsilon_R \langle f_1, h_R^{\varepsilon_1}\rangle \langle f_2, h_R^{\varepsilon_2}\rangle \frac{h_R^{\varepsilon_3}}{\sqrt{|R|}} , \qquad \epsilon_R \in \{-1, +1\}$$

we suppress the dependence of this operator on the three indices $\varepsilon_j \in \{0,1\}^{\vec{d}}$, as well as the choices of signs ϵ_R.

5.1. **Theorem.** *Recall that* $\vec{d} = (d_1, \dots, d_t)$ *and that* $\varepsilon_j = (\varepsilon_{j,1}, \dots, \varepsilon_{j,t})$*. If for all* $1 \le s \le t$*, there is at most one choice of* $j = 1, 2, 3$ *with* $\varepsilon_{j,s} = \vec{1}$*, then the operator* B *satisfies*

$$\mathrm{B} \,:\, L^p \times L^q \longrightarrow L^r , \qquad 1 < p, q < \infty ,\ \tfrac{1}{p} + \tfrac{1}{q} = \tfrac{1}{r} .$$

If in addition, $\varepsilon_1 \neq \vec{1}$*, we will have the estimates*

$$\mathrm{B} \,:\, \mathrm{BMO} \times L^p \longrightarrow L^p\,, \qquad 1 < p < \infty\,.$$

6. Discrete Analogs of Riesz Transforms

The Riesz transforms can be obtained from certain types of linear operators which map Haar functions to themselves (Haar transforms), a fact demonstrated by Petermichl [18] for the Hilbert transform, and Petermichl, Treil and Volberg [19] for Riesz transforms.

Let us describe the Haar transforms we consider. Fix a dimension d, and a choice of d-dimensional dyadic grid $\mathcal{D}$. Let $\sigma : \mathcal{D} \to \mathcal{D}$ such that $2^d|\sigma(I)| = |I|$, for all $I \in \mathcal{D}$. Use the same notation for a map

$$\sigma \,:\, \mathrm{Sig} \longrightarrow \mathrm{Sig} \cup \{0\}$$

where $\mathrm{Sig} \stackrel{\mathrm{def}}{=} \{0,1\}^d - \{\vec{1}\}$ is the set of 'signatures' associated with the wavelets. If $\sigma(\epsilon) = 0$ then $h^{\sigma(\epsilon)} := 0$.

Define Q by

$$\mathrm{Q}\, h_I^{\varepsilon} \stackrel{\mathrm{def}}{=} h_{\sigma(I)}^{\sigma(\varepsilon)}, \tag{6.1}$$

and then extending linearly. There are two facts that we need about these transforms. See [19] for the proof of this proposition.

6.2. **Proposition.** *(a) The operators* Q *as defined above map* $L^p(\mathbb{R}^d)$ *into itself for all* $1 < p < \infty$.
(b) The Riesz transforms are in the convex hull of the class of operators Q*, the convex hull taken with respect to the strong operator topology.*

We need tensor products of the operators Q we have just described. For $\vec{d} = (d_1, d_2, \ldots, d_t)$, and $1 \leq s \leq t$, let Q_s be an operator as above, acting on $L^2(\mathbb{R}^{d_s})$. We will use the same notation for the operator on $L^2(\mathbb{R}^{\vec{d}})$ that equals Q_s on the $L^2(\mathbb{R}^{d_s})$, and is the identity on the orthocomplement of $L^2(\mathbb{R}^{d_s})$ in $L^2(\mathbb{R}^{\vec{d}})$.

6.3. **Proposition.** *The operators*

$$\vec{\mathrm{Q}} \stackrel{\mathrm{def}}{=} \mathrm{Q}_1 \otimes \cdots \otimes \mathrm{Q}_t \tag{6.4}$$

extend to bounded linear operators from $L^p(\mathbb{R}^{\vec{d}})$ *to itself for all* $1 < p < \infty$.

Proof of Proposition 6.3. The proof follows by duality. Suppose $1 < p < \infty$ and $f \in L^p(\mathbb{R}^{\vec{d}}) \cap L^2(\mathbb{R}^{\vec{d}})$ and $g \in L^{p'}(\mathbb{R}^{\vec{d}})$. Computation gives

$$\begin{aligned} |\langle \vec{\mathrm{Q}} f, g\rangle| &= \sum_{\vec{\varepsilon}\in \mathrm{Sig}_{\vec{d}}} \sum_{R\in\mathcal{D}_{\vec{d}}} \langle f, h_R^{\vec{\varepsilon}}\rangle \langle g, h_{\sigma(R)}^{\sigma(\vec{\varepsilon})}\rangle \\ &\le \int_{\mathbb{R}^{\vec{d}}} \mathrm{S}_{\vec{d}}(f)(x) \mathrm{S}_{\vec{d}}(g)(x) dx \end{aligned}$$

where we have defined the square function

$$\mathrm{S}_{\vec{d}}(f)(x) \stackrel{\mathrm{def}}{=} \left(\sum_{\vec{\varepsilon}\in \mathrm{Sig}_{\vec{d}}} \sum_{R\in\mathcal{D}_{\vec{d}}} |\langle f, h_R^{\vec{\varepsilon}}\rangle|^2 \frac{\mathbf{1}_R(x)}{|R|} \right)^{1/2} .$$

Using that

$$\mathrm{Haar}_{\mathcal{D}_{\vec{d}}} \stackrel{\mathrm{def}}{=} \left\{ h_R^{\vec{\varepsilon}} \,:\, R \in \mathcal{D}_{\vec{d}},\ \vec{\varepsilon} \in \mathrm{Sig}_{\vec{d}} \right\}.$$

is a basis for $L^p(\mathbb{R}^{\vec{d}})$ it is straightforward to show that $\|\mathrm{S}_{\vec{d}}(f)\|_p$ is an equivalent norm for $L^p(\mathbb{R}^{\vec{d}})$ when $1 < p < \infty$. This and an application of Cauchy-Schwarz implies

$$|\langle \vec{\mathrm{Q}} f, g\rangle| \lesssim \|f\|_p \|g\|_{p'}.$$

The above inequality and the density of $L^2(\mathbb{R}^{\vec{d}}) \cap L^p(\mathbb{R}^{\vec{d}})$ in $L^p(\mathbb{R}^{\vec{d}})$ gives that $\vec{\mathrm{Q}}$ is a bounded operator. □

7. Proof of Upper Bound for Commutators

To prove our main Theorem, it suffices to consider the commutators in which the Riesz transforms are replaced by choices of the Haar transforms as given in (6.1).

Define commutators by

$$\mathrm{C}_{\vec{\mathrm{Q}}}(b, f) \stackrel{\mathrm{def}}{=} [\cdots [\mathrm{M}_b, \mathrm{Q}_1], \cdots, \mathrm{Q}_t] \tag{7.1}$$

where the Q_s are given as in (6.4), and we view the commutator above as acting on $L^p(\mathbb{R}^{\vec{d}})$. In the remainder of this section we prove this proposition.

7.2. **Proposition.** *The commutators* $\mathrm{C}_{\vec{\mathrm{Q}}}$ *map* $\mathrm{BMO} \times L^p$ *into* L^p *for* $1 < p < \infty$.

We make some detailed remarks about the one parameter case. Let Q be as in (6.1), and consider the commutator

$$[\mathrm{M}_{h_{I'}^{\varepsilon'}}, \mathrm{Q}] h_I^{\varepsilon} = h_{I'}^{\varepsilon'} h_{\sigma(I)}^{\sigma(\varepsilon)} - \mathrm{Q}\, h_{I'}^{\varepsilon'} h_I^{\varepsilon} \tag{7.3}$$

There is no contribution if $I \cap I' = \emptyset$. The other cases yield what follows:

$$[\mathrm{M}_{h_{I'}^{\varepsilon'}}, \mathrm{Q}]h_I^\varepsilon = \begin{cases} 0 & I \subsetneq I' \\ \pm|I|^{-1/2} h_{\sigma(I)}^{\sigma(\varepsilon)} - \mathrm{Q}(h_I^{\varepsilon'} h_I^{\epsilon}) & I = I' \\ h_{\sigma(I)}^{\varepsilon'} h_{\sigma(I)}^{\sigma(\varepsilon)} \pm |I|^{-1/2} h_{\sigma^2(I)}^{\sigma(\varepsilon')} & I' = \sigma(I) \\ \pm|I|^{-1/2} h_{\sigma(I')}^{\sigma(\varepsilon')} & I' \subsetneq I \text{ and } I' \cap \sigma(I) = \emptyset \\ \pm|\sigma(I)|^{-1/2} h_{I'}^{\varepsilon'} \pm |I|^{-1/2} h_{\sigma(I')}^{\sigma(\varepsilon')} & I' \subsetneq \sigma(I)\,. \end{cases} \tag{7.4}$$

The first triangular sum corresponding to $I \subsetneq I'$ is therefore trivial. It illustrates the essential cancellation contained in the commutator. It remains to consider the diagonal sums $I = I'$ and $\sigma(I) = I'$ as well as the other triangular sum $I' \subsetneq \sigma(I)$. All these do not require any cancellation of the commutator as we shall see.

First, we calculate the diagonal part for $I = I'$ explicitly and obtain the two sums:

$$\sum_I \sum_{\varepsilon, \varepsilon' \neq \vec{1}} \pm (b, h_I^{\varepsilon'})(f, h_I^\varepsilon)|I|^{-1/2} Q(h_I^\varepsilon) \;\text{ and }\; Q\left(\sum_I \sum_{\varepsilon,\varepsilon' \neq \vec{1}} (b, h_I^{\varepsilon'})(f, h_I^\varepsilon)|I|^{-1/2} h_I^{\tilde{\varepsilon}}\right)$$

The first sum is essentially a finite combination of operators of the form $B(b, Qf)$ and hence bounded. Here we disregard the sign changes as the paraproducts are unconditionally convergent. We also note the fact that we may change the signatures of the Haar functions, which is included in our definition of paraproduct. In the same sense, the second sum has terms of the form $Q(B(b, f))$. The term $h_I^{\tilde{\varepsilon}}$ stems from the product of two Haar functions based on I. The signature $\tilde{\varepsilon}$ may be equal to $\vec{1}$, in which case we still have a convergent paraproduct.

For the other diagonal sum, which corresponds to $I' = \sigma(I)$, the explicit calculation and the reasoning is similar.

At last, we demonstrate the triangular sum by explicit calculation. We start by summing the first summand over I:

$$\sum_{I'} \sum_{\varepsilon' \neq \vec{1}} (b, h_{I'}^{\varepsilon'}) h_{I'}^{\varepsilon'} \sum_{I : I \supsetneq I'} \sum_{\varepsilon \neq \vec{1}} (f, h_I^\varepsilon) Q(h_I^\varepsilon)$$

The inner part is equal to $\sum_{I : I \supsetneq I'} \sum_{\varepsilon \neq \vec{1}} (Qf, h_I^\varepsilon) h_I^\varepsilon$ which on I' equals a renormalized average of Qf, namely $(Qf, h_{I'}^{\vec{1}})|I'|^{-1/2}$. We conclude that the first sum is a combination of paraproducts of the form $B(b, Qf)$. Similarly, the second sum has terms of the form $QB(b, f)$ with an indicator falling on f.

This proves that our commutator is a finite linear combination of terms of the form

$$\mathrm{Q}\,\mathrm{B}(b, f), \qquad \mathrm{B}(b, \mathrm{Q}\, f)$$

for appropriate choices of Q and paraproducts B. It is essential to note that these paraproducts are of the form that we can permit $b \in \mathrm{BMO}$, so that the one parameter form of Theorem 5.1 applies to prove Proposition 7.2 in this case. This completes the discussion in the one parameter case.

In passing to the multi-parameter paraproducts, we adopt the notation of Proposition 6.3. Thus, we have a choice of $\vec{d} = (d_1, \dots, d_t)$ dimensional dyadic rectangles $\mathcal{D}_{\vec{d}}$, and a choice of operators Q_s, for $1 \le s \le t$, which is an operator as in (6.1) acting on $\mathbb{R}^{d_s}$. We can then make an explicit computation of the commutator as in (7.3), namely

$$[\cdots [\mathrm{M}_{h_R^{\vec{\varepsilon}}}, \mathrm{Q}_1], \cdots \mathrm{Q}_t]$$

The result is a tensor product of the results on the right hand side of (7.4). In other words, the multi parameter commutator splits into its components.

It follows that we can write the commutator as a finite linear combination of terms

$$\vec{\mathrm{Q}}\,\mathrm{B}(b, f)\,, \qquad \mathrm{B}(b, \vec{\mathrm{Q}} f)$$

for different choices of multi-parameter paraproduct B and different choices of operator $\vec{\mathrm{Q}}$. Thus Proposition 7.2 follows from Theorem 5.1.

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